# Exact results on massless 3-flavor QCD through new anomaly matching

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References: 1710.08923[hep-th], 1711.10487[hep-th]

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## Introduction: Anomaly matching condition

# Quantum field theory

Quantum field theory is an important and ubiquitous tool to study low-energy behaviors of quantum many-body systems.

- Nuclear and hadron physics
  $\Rightarrow$ Quantum chromodynamics
- Pion, Kaon physics
  $\Rightarrow$ Nonlinear sigma models
- Quantum Hall effects, Topological states of matters
  $\Rightarrow$ Topological field theories

# Symmetry, Topology

Solving QFT is usually very hard.

Example QCD contains quarks and gluons as elementary fields, but it explains physics of hadrons.

## Question

*Can we say something rigorous even when QFT of our interest is strongly coupled?*

Keywords Symmetry and Topology

Especially, we pay attention to anomaly.

# 't Hooft anomaly

't Hooft anomaly
= Symmetry of the quantum theory that can't be gauged

Example Consider the 4d free massless Dirac fermion

$$S = \int \mathrm{d}^4x\, \overline{\psi}\gamma_\mu \partial_\mu \psi$$

This quantum theory has the $U(1)_{\text{vector}} \times U(1)_{\text{axial}}$ symmetry. Once one gauges the $U(1)_{\text{vector}}$ symmetry, $U(1)_{\text{axial}}$ is explicitly broken,

$$\partial_\mu j_{5\mu} = \frac{1}{8\pi^2} F_{\mu\nu}\widetilde{F}_{\mu\nu}.$$

One cannot gauge the $U(1)_{\text{vector}} \times U(1)_{\text{axial}}$ symmetry.

# Anomaly matching

**Theorem**

*An 't Hooft anomaly is renormalization-group invariant.*

Once the 't Hooft anomaly is computed in UV, the same anomaly must be reproduced in IR.

**Example**

Massless QCD has symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ with an 't Hooft anomaly:

$$D^\mu J^a_\mu = 0, \quad D^\mu J^{5a}_\mu = \frac{1}{4\pi^2}\mathrm{tr}\left(T^a\left\{F_V^2 + \frac{1}{3}F_A^2 + \cdots\right\}\right)$$

$\Rightarrow$ In the hadron picture, it is matched by Chiral symmetry breaking and the WZW term for the pions.

# What I will do

- Explain the new 't Hooft anomaly related to "center" and discrete axial symmetry
- Apply its constraint on the chiral symmetry breaking at finite densities
- Apply its constraint to the phase structure of $\mathbb{Z}_3$-QCD

(Related works:
Gaiotto, Kapustin, Komargodski, Seiberg, 1703.00501 about $\theta$ vacua of Yang-Mills theory,
YT, Kikuchi, 1705.01949 about Gauge theory with bifundamental quarks,
Komargodski, Sharon, Thorngren, Zhou, 1705.04786 about Abelian Higgs model, ...)

## Calculation: New 't Hooft anomaly of massless 3-flavor QCD

# Symmetry of massless QCD

QCD Lagrangian:

$$S = \frac{1}{2g^2}\int \mathrm{tr}(G \wedge *G) + \int \sum_{f=1}^{N_F} \overline{q}_f \gamma_\mu D_\mu q.$$

Symmetry of massless QCD (i.e. Lagrangian + $\mathcal{D}\overline{q}\mathcal{D}q$):

$$\frac{SU(N_F)_L \times SU(N_F)_R \times U(1)_V \times (\mathbb{Z}_{2N_F})_A}{\mathbb{Z}_{N_C} \times (\mathbb{Z}_{N_F})_L \times (\mathbb{Z}_{N_F})_R \times \mathbb{Z}_2}$$

- For later purpose, I evade the double counting correctly (e.g. $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$, $U(1)_L \times U(1)_R = [U(1)_V \times U(1)_A]/\mathbb{Z}_2$)
- Due to the gauge invariance, symmetry must be divided by $\mathbb{Z}_{N_C} \subset SU(N_C)$.

# Further comments on symmetry

One way to think about $(\mathbb{Z}_{2N_F})_A$ is that it is anomaly-free subgroup of $U(1)_A$.

For later application, let us see another perspective:

$$(\mathbb{Z}_{2N_F})_A \subset SU(N_F)_L \times SU(N_F)_R \times U(1)_V.$$

Indeed, the generator of $(\mathbb{Z}_{2N_F})_A$ can be written as

$$\mathrm{e}^{\frac{2\pi}{2N_F}\mathrm{i}\gamma_5}\mathbf{1}_{N_F} = \exp\left(\frac{2\pi}{N_F}\mathrm{i}\frac{1+\gamma_5}{2}\mathrm{diag}[1,\ldots,1,1-N_F]\right)\mathrm{e}^{-2\pi\mathrm{i}/(2N)}$$

Quick message: SSB of $(\mathbb{Z}_{2N_F})_A$ gives a stronger constraint on ChSB than that of continuous chiral symmetry.

# Gauging the vector symmetry 1

We pay attention to the vector-like symmetry,

$$\frac{SU(N_F)_V \times U(1)_V}{(\mathbb{Z}_{N_C}) \times (\mathbb{Z}_{N_F})},$$

and $(\mathbb{Z}_{2N})_{\text{axial}}$.

To detect the 't Hooft anomaly, we introduce the background gauge fields for the vector-like symmetry;

- $SU(N_F)_V$ one-form gauge field: $A_f$
- $U(1)_V$ one-form gauge field: $A_q$
- $(\mathbb{Z}_{N_C})$ two-form gauge field: $B_c$ $(N_C B_c = \mathrm{d}C_c)$
- $(\mathbb{Z}_{N_F})$ two-form gauge field: $B_f$ $(N_F B_f = \mathrm{d}C_f)$

# Gauging the vector symmetry 2

Introducing the two-form gauge fields $B_{\rm c}$ and $B_{\rm f}$, the $SU(N_{C,F})$ gauge connections are once promoted to $U(N_{C,F})$ gauge connections (Kapustin, Seiberg, 2014),

$$\widetilde{a} = a + \frac{1}{N_C} C_{\rm c}, \quad \widetilde{A_{\rm f}} = A_{\rm f} + \frac{1}{N_F} C_{\rm f}.$$

The covariant derivative and field strength are replaced as

$$\begin{aligned} D(\widetilde{a}, \widetilde{A_{\rm f}}, A_{\rm q}) q &= (\mathrm{d} + \mathrm{i}\widetilde{a} - \mathrm{i}(\widetilde{A_{\rm f}})^t + \mathrm{i}A_{\rm q}) q, \\ \widetilde{G} + B_c &= (\mathrm{d}\widetilde{a} + \mathrm{i}\widetilde{a} \wedge \widetilde{a}) + B_c, \end{aligned}$$

by the minimal coupling procedure.

We denote the partition function after this process as $\mathcal{Z}[(A_{\rm f}, A_{\rm q}, B_{\rm c}, B_{\rm f})]$.

# Mixed 't Hooft anomaly for massless QCD

$(\mathbb{Z}_{2N})_{\text{axial}}$ can change $\mathcal{Z}$ as

$$\mathcal{Z}[(A_{\text{f}}, A_{\text{q}}, B_{\text{c}}, B_{\text{f}})] \mapsto \mathcal{Z}[(A_{\text{f}}, A_{\text{q}}, B_{\text{c}}, B_{\text{f}})] \exp(\mathrm{i}\mathcal{A}).$$

The 't Hooft anomaly $\mathcal{A}$ is given by Fujikawa method:

$$\mathcal{A} = 2\frac{2\pi}{2N}\frac{1}{8\pi^2}\int \mathrm{tr}_{\text{c,f}}\left[F \wedge F\right],$$

with $F = \mathrm{i}^{-1} D(\widetilde{a}, \widetilde{A_{\text{f}}}, A_{\text{q}}) \wedge D(\widetilde{a}, \widetilde{A_{\text{f}}}, A_{\text{q}})$.

For simplicity of expression, let us set $N_C = N_F = N$, then

$$\mathcal{A} = -\frac{N}{2\pi}\int B_{\text{c}} \wedge B_{\text{f}}.$$

After gauging the vector symmetry correctly, $(\mathbb{Z}_{2N})_A$ is broken by $\mathcal{A}$.

# Summary of computation

We show that massless $N$-flavor QCD has a mixed 't Hooft anomaly between

$$\frac{SU(N_F)_V \times U(1)_V}{(\mathbb{Z}_{N_C}) \times (\mathbb{Z}_{N_F})} \quad \text{and} \quad (\mathbb{Z}_{2N})_{\text{axial}}.$$

The anomaly is given as $(N_C = N_F = N)$

$$\mathcal{Z}[(A_{\text{f}}, A_{\text{q}}, B_{\text{c}}, B_{\text{f}})] \mapsto \mathcal{Z}[(A_{\text{f}}, A_{\text{q}}, B_{\text{c}}, B_{\text{f}})] \exp\left(-\frac{\mathrm{i}N}{2\pi}\int B_{\text{c}} \wedge B_{\text{f}}\right).$$

We will discuss consequences of this anomaly.

## Application 1: Constraint on the zero-temperature phase at finite $\mu$

# Anomaly at finite density

In our derivation, we can introduce $\mu$ without affecting the discussion, because we can set $A_{q0} = \mathrm{i}\mu$.

The quick result is that $H_{QCD} - \mu Q$ cannot have the trivially gapped ground state:

- SSB of $SU(N)_V \times U(1)_V$ or $(\mathbb{Z}_{2N})_A$
- Conformal behavior
- Topological order

For $N > 2$, we can give a more detailed claim for the first option:

- SSB of $SU(N)_V$ or $(\mathbb{Z}_{2N})_A$

i.e. SSB of $U(1)_V$ is not enough for matching the anomaly.

# SSB of chiral symmetry

The usual chiral symmetry breaking is realized by $\langle \bar{q}q \rangle \neq 0$, which breaks $SU(N)_L \times SU(N)_R \to SU(V)$.

In principle, another SSB is possible:

$$SU(N)_L \times SU(N)_R \to SU(N)_V \times (\mathbb{Z}_{2N})_A$$

This is characterized by four-quark condensates, such as

$$\langle \mathrm{tr_f}(T^a \bar{q}(1-\gamma_5)q)\mathrm{tr_f}(T^a \bar{q}(1+\gamma_5)q)\rangle.$$

This vacuum is $(\mathbb{Z}_{2N})_A$ invariant but breaks continuous chiral symmetry (Stern, 1998).

At $\mu = 0$, this is ruled out by QCD inequality. (Kogan, Kovner, Shifman, 1998) But, nothing rules out it at finite $\mu$ so far.

# Constraint on chiral symmetry breaking

(Caution This part is on-going work, and still need checks)

The Nambu-Goldstone bosons in Stern phase is characterized by the following coset field:

$$U \in [(SU(N)_L \times SU(N)_R)/(SU(N)_V \times (\mathbb{Z}_{2N})_A)].$$

By definition, $U \sim \mathrm{e}^{\mathrm{i}\pi^a T^a/f_\pi}$ does not feel the discrete axial symmetry, and thus those pions cannot reproduce our new anomaly $\mathcal{A}$.

$\Rightarrow$ If Stern phase is realized at finite density, additional massless fields must exist to match the anomaly.

## Application 2: Constraint on the phase diagram of massless $\mathbb{Z}_3$-QCD

# $\mathbb{Z}_N$-QCD

We put the boundary condition on the quark field as $(f = 1, \ldots, N)$

$$q_f(\boldsymbol{x}, x_4 + L) = \exp\left(\frac{2\pi \mathrm{i} f}{N}\right) q_f(\boldsymbol{x}, x_4) = \omega^f q_f(\boldsymbol{x}, x_f).$$

This theory is called $\mathbb{Z}_N$-QCD (Kouno et al. 2012, ...)
This symmetry-twisted partition function is the thermal expectation value of the symmetry-twisting operator;

$$\mathcal{Z}_\Omega = \mathrm{tr}_{\mathcal{H}} \left[ \mathrm{e}^{-L(\widehat{H} - \mu \widehat{Q})} \exp\left( \mathrm{i} \sum_{f=1}^{N} \frac{2\pi f}{N} \widehat{Q}_f \right) \right].$$

We will see that this twisted boundary condition plays an essential role for the existence of 't Hooft anomaly $\mathcal{A}$ in the $S^1$-compactified theory.

# Symmetry of massless $\mathbb{Z}_N$-QCD

There are three symmetries for massless $\mathbb{Z}_N$-QCD:

$$(\mathbb{Z}_N)_{\text{shift}}, \quad \frac{U(1)_F^{N-1} \times U(1)_V}{(\mathbb{Z}_N)_C \times (\mathbb{Z}_N)_F}, \quad \text{and} \quad (\mathbb{Z}_{2N})_{\text{axial}}$$

$(\mathbb{Z}_N)_{\text{shift}}$ appears thanks to the boundary condition, and is defined by

$$\mathrm{tr}_{\mathrm{c}} \left[ P \exp\left( \mathrm{i} \int a \right) \right] \mapsto \omega \, \mathrm{tr}_{\mathrm{c}} \left[ P \exp\left( \mathrm{i} \int a \right) \right], \; q_f \mapsto q_{f+1}.$$

Since the boundary condition can be regarded as a Polyakov loop, it is also changed as

$$q_f(\boldsymbol{x}, x_4 + L) = \omega^{f+1} q_f(\boldsymbol{x}, x_f).$$

Performing $q_f \mapsto q_{f+1}$, we recover the original boundary condition, and thus it is a symmetry.

# Mixed 't Hooft anomaly of massless $\mathbb{Z}_N$-QCD

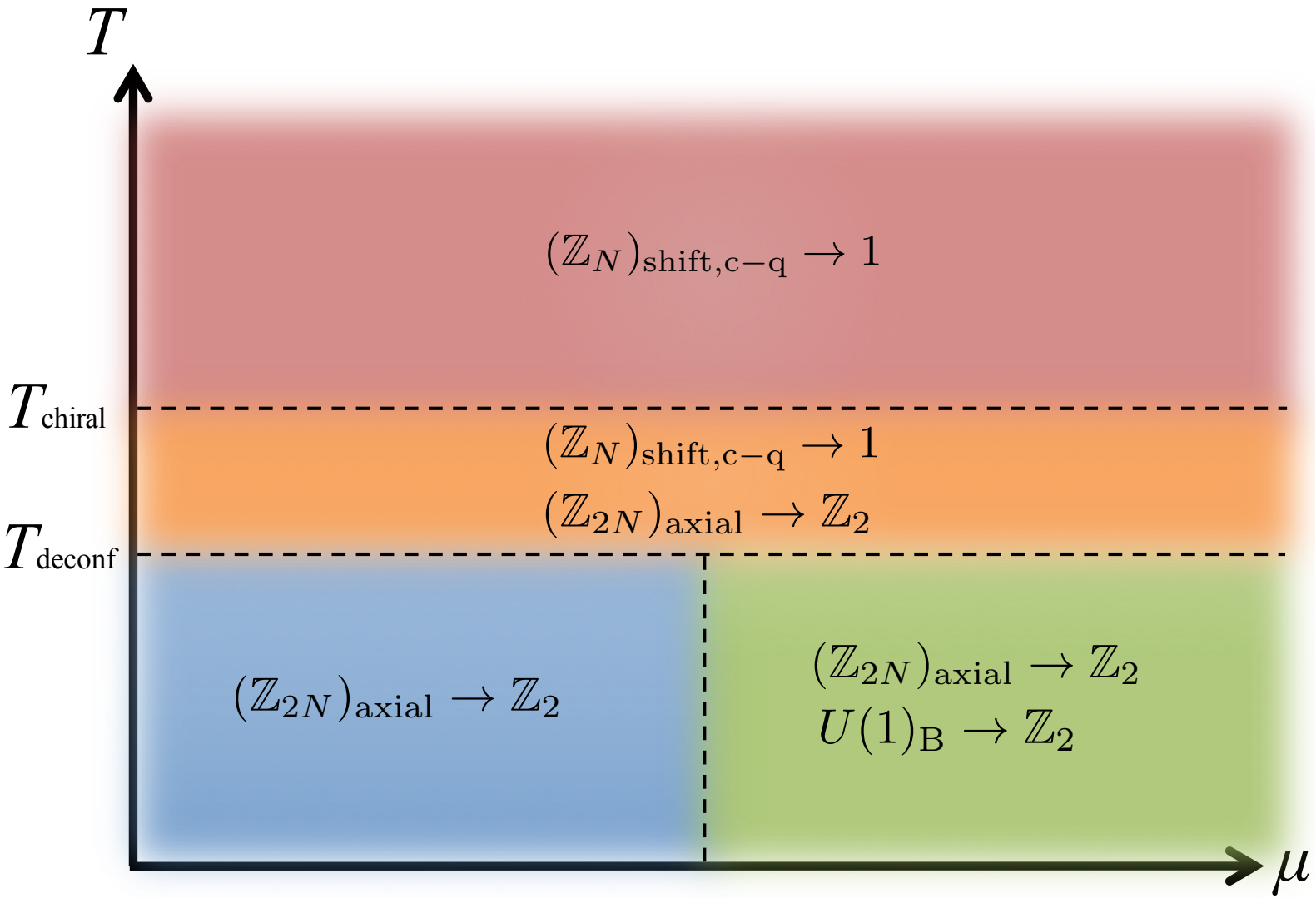
We can gauge $(\mathbb{Z}_N)_{\text{shift}}$ and $(U(1)_{\text{f}}^{N-1} \times U(1)_{\text{q}})/((\mathbb{Z}_N)_{\text{c-q}} \times (\mathbb{Z}_N)_{\text{f-q}})$ by introducing the following two-form gauge fields ($+$ ordinary ones):

$$B_{\text{c}} = B_{\text{c}}^{(1)} \wedge L^{-1} \mathrm{d}x^4 + B_{\text{c}}^{(2)}, \quad B_{\text{f}} = B_{\text{f}}^{(2)}.$$

Substituting it into the four-dimensional anomaly, we obtain the anomaly for massless $\mathbb{Z}_N$-QCD:

$$\mathcal{A} = -\frac{N}{2\pi} \int B_{\text{c}}^{(1)} \wedge B_{\text{f}}^{(2)} \in \frac{2\pi}{N} \mathbb{Z}.$$

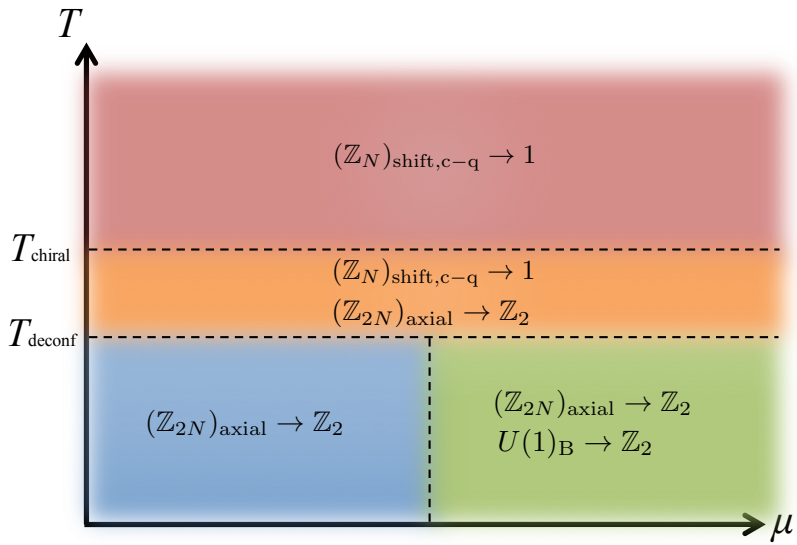
# One of Possible phase diagrams



# Comment on Possible phase diagrams

We have a mixed 't Hooft anomaly between $(\mathbb{Z}_N)_{\rm shift}$, $\frac{U(1)_F^{N-1}\times U(1)_V}{(\mathbb{Z}_N)_C\times(\mathbb{Z}_N)_F}$, and $(\mathbb{Z}_{2N})_{\rm axial}$.

1. System is conformal or breaks some of symmetries
2. When anomaly is matched by SSB, $T_{\rm deconf} \le T_{\rm chiral}$ if flavor is unbroken.
3. SSB of $U(1)_V$ is not enough to match the anomaly.




Note: These constraints are not satisfied by PNJL model. Ginzburg-Landau description does not respect anomaly matching, so we must go beyond!

# Summary

- We find a new 't Hooft anomaly of massless QCD.
  Anomaly matching gives nonperturbative constraints on low-energy physics.
- The constraint is meaningful at finite $\mu$ and zero temperature:
  Naive Stern phase seems to be ruled out
- It is also meaningful for $\mathbb{Z}_N$-QCD:
  The phase is always non-trivial.